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Order Reduction of Linear-Quadratic-Gaussian-Designed Controllers

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Linear-quadratic-Gaussian (LQG) procedures for controller design yield compensators of the same order as the plant. However, practical design considerations often dictate low-order controllers. This paper presents a procedure for reducing the order of LQG-designed controllers. The proposed method is based on the same performance index J used for designing the optimal regulator; the value of J for the system with the reduced-order controller is essentially identical to the value obtained with the full-order controller. It is shown that the observer normal coordinates play an important role in the proposed method. The numerical results presented illustrate the effectiveness of this method.

Introduction

CONTROLLER design using linear-quadratic-Gaussian (LQG) procedures yields compensators of the same order as the plant. Many physical systems, such as aeroservoelastic systems, involve a large number of states and therefore result in high-order controllers. These high-order controllers are difficult to implement and may be susceptible to reliability problems. Therefore, it is imperative to find ways to reduce the order of these controllers.

There are two basic approaches used to find low-order controllers: The first approach involves the approximation of the high-order system by a lower order one,[1-3] and then application of the controller design procedure. In the second approach, the controller for the full-order system is first designed, and only then are attempts made to reduce the controller order.[4-7] The method proposed in this paper falls within the latter approach, and deals with a procedure for reducing the LQG-designed high-order controller into a lower order one. It differs from most other existing methods in that no attempts are made to balance the system in any form or to minimize any norm associated with the controller or the plant. The method may be reminiscent of the component cost analysis (CCA) method,[8,9] but as will be shown later, it differs widely from it.

Description of the Method

General

The method proposed assumes that the optimal control input u has been determined from optimal regulator theory, such that it minimizes a given quadratic performance index J. It is then assumed that all of the states, except for a single state, are used for feedback in the plant-regulator system, and J and eigenvalues for the closed-loop system are computed. This procedure is repeated, with a different single state omitted each time from the optimal feedback expression, until all of the states have been scanned in such a manner, and the resulting effects on J and on the eigenvalues are computed. Because J is minimal with full-state feedback, it is expected that the omission of a single state from the optimal feedback input will give rise to higher values of J. In this way, a numerical value can be attached to the importance of each state in its effect on the value of J. Those states that affect J the most will be considered the most important states, whereas those states that have a negligible effect on J will be considered unimportant. Our objective will be to feed back as few states as possible with negligible effects on the value of J as obtained for the full-state feedback case. As will be shown, the identification of these important regulator states is not sufficient to ensure the desired closed-loop performance of the reduced controller. This is true because the important states so determined cannot be measured directly, and therefore, they need to be reconstructed using Kalman filter theory. This reconstruction process will be shown to have a profound effect on the procedure just described.

Before embarking on the mathematical formulation of the proposed method, it is appropriate to mention the three main differences that exist between this method and the CCA method, which is somewhat related. First, unlike the CCA method, no decomposition of J in terms of the contributions of the different states to its numerical value is attempted in the proposed method. It is this decomposition of J into state components that gives rise to a basic difficulty that is inherent in the CCA method. States with a small component in J are considered by the CCA method as unimportant for feedback, while at the same time, both the value of J and its state component values are determined using full-state feedback. It can readily be argued that the small components in J of specific states is a direct result of the full-state feedback used during the computation and the decomposition of J, and that in the absence of the feedback associated with these "unimportant" states, J may assume widely different values. Second, J generally consists of two quadratic expressions: one involving plant states only, the other involving observer states only. The controller-reduction algorithm proposed by the CCA method relates only to the quadratic expression involving the observer states.[8] Clearly, any reduction in the order of the controller affects both quadratic expressions simultaneously, and the coupling between them cannot be disregarded. Third, the CCA method does not allow for the possibility where a specific state, although truly unimportant in itself, has a significant effect on the reconstructed values of the important states through the cross-coupling terms present in the observer equation. This specific point will be shown to be of extreme importance during the formulation of the proposed method.

With these general remarks made, a mathematical formulation of the proposed method will next be presented.

Mathematical Formulation

Let the system to be controlled be linear and time invariant, given in the following state space-form:

$$\dot{x} = Ax + Bu + Gw$$
$$y = Cx + v \tag{1}$$

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where A is an $n \times n$ plant matrix, B is an $n \times l$ input matrix, C is an $m \times n$ output matrix, and v and w are zero mean, uncorrelated, white-noise processes with intensities $V > 0$ and $W \geq 0$. Let the steady-state performance index J be defined by

$$J = \lim_{t \to \infty} E(x^T Q x + u^T R u) \tag{2}$$

where $Q \geq 0$ and $R > 0$.

Following optimal control theory, the input u that minimizes J is given by

$$u = -Fx \tag{3}$$

where F is an $l \times n$ feedback gain matrix obtained through the solution of the usual algebraic Riccati equations.

Substituting Eq. (3) into Eq. (1) yields

$$\begin{aligned} \dot{x} &= (A - BF)x + Gw \\ y &= Cx + v \end{aligned} \tag{4}$$

As stated earlier, a natural way to determine which states in Eq. (3) are important would be to set each column of F, in turn, to zero and evaluate the performance index by solving the Lyapunov equation

$$(A - BF)X + X(A - BF)^T + GWG^T = 0 \tag{5}$$

and then evaluating J from the following equation:

$$J = \mathrm{tr}(QX) + \mathrm{tr}(F^T RFX) \tag{6}$$

Unfortunately, we cannot generally measure the states directly and instead must use the Kalman filter to obtain the reconstruction of the states $\hat{x}$ to implement the feedback control with the reconstructed states; that is,

$$u = -F\hat{x} \tag{7}$$

The reconstructed states are obtained from the following observer equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{8}$$

where L is an $n \times m$ matrix obtained from optimal observer theory through the solution of the usual Riccati equations. The output $\hat{y}$ is defined by

$$\hat{y} = C\hat{x} \tag{9}$$

Substituting Eqs. (7) and (9) into Eq. (8) yields the following observer equation:

$$\dot{\hat{x}} = (A - BF - LC)\hat{x} + Ly \tag{10}$$

In general, the observer equation as given by Eq. (10) is fully coupled, and therefore the possibility exists for the unimportant reconstructed states to strongly affect the values of the important reconstructed states through the cross-coupling terms. An example demonstrating this effect will be presented later. It is therefore essential to decouple the important reconstructed states from the unimportant reconstructed states using a coordinate transformation matrix T. If this coordinate transformation is applied to the previously determined important and unimportant regulator states, it introduces couplings between them, thus confusing their relative importance.

To overcome this difficulty it is necessary to start the procedure by first decoupling the observer states in Eq. (10). This can be achieved by letting T consist of the eigenvectors of the observer equation (10). In this way, Eq. (10) can be reduced to a block diagonal form, with coupling existing only within each pair of states that is associated with complex conjugate eigenvectors. That is, let

$$\hat{x} = T\hat{z} \tag{11}$$

and substitute this transformation in Eq. (10) to obtain

$$\dot{\hat{z}} = A_0 \hat{z} + L_T y \tag{12}$$

where

$$\begin{aligned} A_0 &= T^{-1}(A - BF - LC)T \\ L_T &= T^{-1}L \end{aligned} \tag{13}$$

The reconstructed $\hat{z}$ coordinates will also be referred to as observer normal coordinates. To avoid the coupling within the block diagonal form of the observer matrix A_0, the simultaneous effects on J of both states forming the block will be investigated. This leads to either both $\hat{z}$ states (within the block) found to be important, or both states (within the block) found to be unimportant. In this manner, no coupling can possibly exist between important and unimportant reconstructed $\hat{z}$ states.

We now return to the plant-regulator equation (4) and transform the x coordinates into z coordinates using the same transformation matrix T. That is,

$$x = Tz \tag{14}$$

to obtain

$$\dot{z} = T^{-1}(AT - BF_T)z + G_T w \tag{15}$$

where

$$\begin{aligned} F_T &= FT \\ G_T &= T^{-1}G \end{aligned} \tag{16}$$

We now apply our procedure to determine the important z states in the plant-regulator equation (15) by setting, in turn, different columns of F_T (instead of F, as previously done) to zero. This is done by either setting to zero each column of F_T separately, or two columns simultaneously whenever the $\hat{z}$ states relate to a pair of complex conjugate eigenvalues (and eigenvectors). The transformed value of Eq. (2) for the performance index J assumes the form

$$J = \lim_{t \to \infty} E[z^T(T^T QT + F_T^T RF_T)z] \tag{17}$$

Once the important z_i states are determined, the block diagonal observer equation (12) for the reconstruction of these states and the feedback gain matrix F_T are rearranged and partitioned to include all of the elements associated with the important $\hat{z}_i$ states within the partition subscripted by i, and all unimportant $\hat{z}_n$ states within the partition subscripted by n. That is,

$$\begin{Bmatrix} \dot{\hat{z}}_i \\ \dot{\hat{z}}_n \end{Bmatrix} = \begin{bmatrix} A_{0,i} & 0 \\ 0 & A_{0,n} \end{bmatrix} \begin{Bmatrix} \hat{z}_i \\ \hat{z}_n \end{Bmatrix} + \begin{bmatrix} L_{T,i} \\ L_{T,n} \end{bmatrix} y \tag{18}$$

The states $\hat{z}_i$ can be determined from Eq. (18) using either truncation or residualization.[10] If truncation is used, the combined plant-observer (with reduced-order observer) equations can be written by

$$\begin{Bmatrix} \dot{x} \\ \dot{\hat{z}}_i \end{Bmatrix} = \begin{bmatrix} A & -BF_{T,i} \\ L_{T,i}C & A_{0,i} \end{bmatrix} \begin{Bmatrix} x \\ \hat{z}_i \end{Bmatrix} + \begin{bmatrix} G & 0 \\ 0 & T_i^{-1} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{19}$$

where

$$F_T = [F_{T,i} \quad F_{T,n}] \tag{20}$$

When residualization is applied to Eq. (18), a correction term needs to be added to A in Eq. (19). This additional term is obtained from the lower partition of Eq. (18) by letting $\dot{\hat{z}}_n = 0$. This leads to

$$\hat{z}_n = -A_{0,n}^{-1} L_{T,n} y$$

After substituting for y from Eq. (1) we obtain

$$\hat{z}_n = -A_{0,n}^{-1} L_{T,n} C x - A_{0,n}^{-1} L_{T,n} v \tag{21}$$

When $\hat{z}_n$ is multiplied by $F_{T,n}$, Eq. (19) becomes, for the residualization case,

$$\begin{Bmatrix} \dot{x} \\ \dot{\hat{z}}_i \end{Bmatrix} = \begin{bmatrix} A + BF_{T,n} A_{0,n}^{-1} L_{T,n} C & -BF_{T,i} \\ L_{T,i} C & A_{0,i} \end{bmatrix} \begin{Bmatrix} x \\ \hat{z}_i \end{Bmatrix} + \begin{bmatrix} G & BF_{T,n} A_{0,n}^{-1} L_{T,n} \\ 0 & T_i^{-1} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{22}$$

Summary of the Reduction Procedure

The reduction procedure is summarized for convenience in the following:

1) Use LQG procedures to determine the regulator feedback gain matrix F that minimizes the performance index J defined by Eq. (2).

2) Use LQG procedures to determine the observer output gain matrix L.

3) Find the eigenvalues and eigenvectors of the observer matrix

$$(A - BF - LC)$$

and form the transformation matrix T from these eigenvectors. When complex (necessarily conjugate) eigenvectors are obtained, use only one eigenvector solution in T by placing the real part of the eigenvector in one column of T, and the imaginary part of the eigenvector in an adjoining column of T. This will maintain T as a real matrix.

4) Form the transformed plant-regulator equation defined by Eqs. (15) and (16).

5) Determine the important z states by setting, in turn, different columns of F_T to zero and solving the Lyapunov equation

$$[T^{-1}(AT - BF_T)]Z + Z[T^{-1}(AT - BF_T)]^T + G_T W G_T^T = 0$$

Note that when the eigenvalues corresponding to the specific states are complex, the two respective columns in F_T must simultaneously be set to zero when evaluating J.

6) Compute J for each of the conditions tested in item 5, using the equation

$$J = \mathrm{tr}(T^T Q T Z) + \mathrm{tr}(F_T^T R F_T Z)$$

7) Rearrange and partition the observer equation and the matrices F_T and L_T to include all of the elements associated with the important $\hat{z}_i$ states in the top partition and evaluate the performance of the closed-loop system with the reduced-order controller using either one of the following two equations: When truncation is used, solve Eq. (19); when residualization is used, solve Eq. (22).

It may be mentioned at this point that the feedback of the important z states only in the plant-regulator system, as defined by the proposed method, ensures that this reduced-order feedback has a negligible effect on J, thus maintaining the regulator in an essentially optimal state. The reconstructed values of these important z states are totally unaffected by the order reduction, and they represent the exact optimal values produced by the Kalman-Bucy optimal filter theory. Hence, based on the principle of separation, the full-order plant with the reduced-order controller yields essentially the same performance index J with the full-order controller.

Numerical Example

The procedure for controller order reduction will be illustrated using the mathematical model[11] of NASA's drone for aerodynamic and structural testing—aerodynamic research wing 1 (DAST-ARW1). This model consists of 10 structural (second-order) modes, which are represented by 20 states. The aerodynamics (for Mach number $M = 0.9$) are represented using Roger's rational approximation[12] involving two lag terms. This leads to an additional 20 states, which will be referred to as aerodynamic states. The aileron is located near the wingtip and is activated by a third-order actuator, which contributes another three states. Finally, the system is assumed to be subjected to atmospheric turbulence, which is modeled by the Dryden gust spectrum. This yields two additional states. We therefore end up with a plant that has a total of 45 states. These states are arranged in the plant equation so that the 20 structural states appear first, followed by the 20 aerodynamic states and the three actuator states. The two gust states appear at the very end. Clearly, any LQG design results in a controller of order 45. Attempts will be made in the following to apply our procedure to this example to arrive at a controller with a lower order.

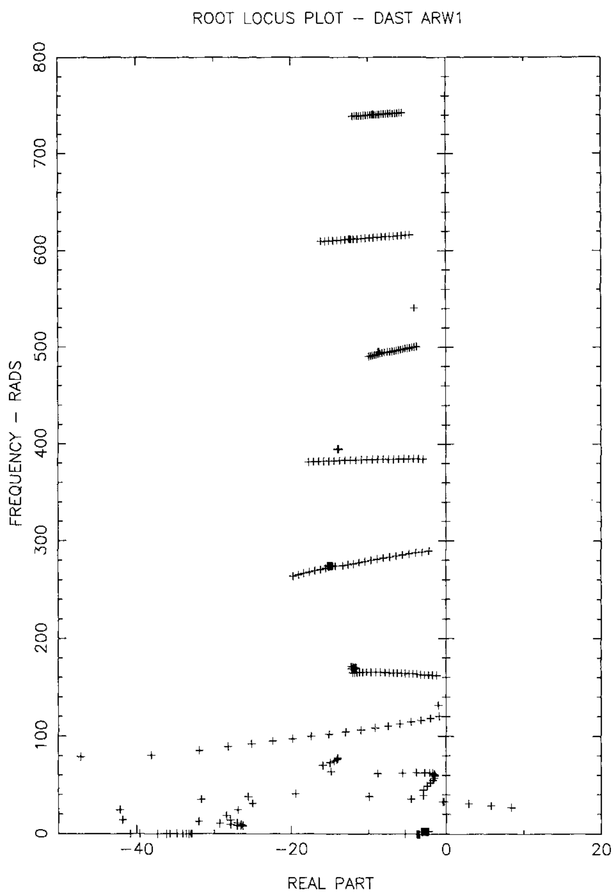

Fig. 1 Closed-loop root locus plot of plant with full-order controller (45th order).

The full-state feedback LQG design was obtained in this example by using the fictitious noise adjustment procedure[13] for robustness improvement. The closed-loop root locus plot with full-state feedback is shown in Fig. 1. The flutter dynamic pressure at $M = 0.9$ is $Q_F = 191$ psf for the closed-loop case. This compares with $Q_F = 96$ psf obtained for the open-loop case. The LQG design point for this example is at a dynamic pressure $Q_D = 156.9$ psf, and the Nyquist plot for the 45th-order controller at the design point is shown in Fig. 2. The minimum singular value of the return difference in this single-input, single-output (SISO) example is $\sigma_{\min} = 0.886$. The control surface activity at the design point, due to a 1-ft/s gust velocity, is given by $\delta = 4.22$ deg, and $\dot{\delta} = 183$ deg/s, with performance index $J = 5484$ (obtained with $R = 10^{-6}$ and $Q = 0$). The LQG design yields the full-state matrices F and L, and therefore, the first two steps of the design procedure have thus been achieved. At this point, no attempt will be made to proceed with step 3 of our procedure. Instead, an attempt will be made in the following section to determine the important x states in the plant-regulator closed-loop system.

Analysis of the Plant-Regulator System

It is expected that the important states to be determined for the closed-loop plant-regulator system will, in general, be

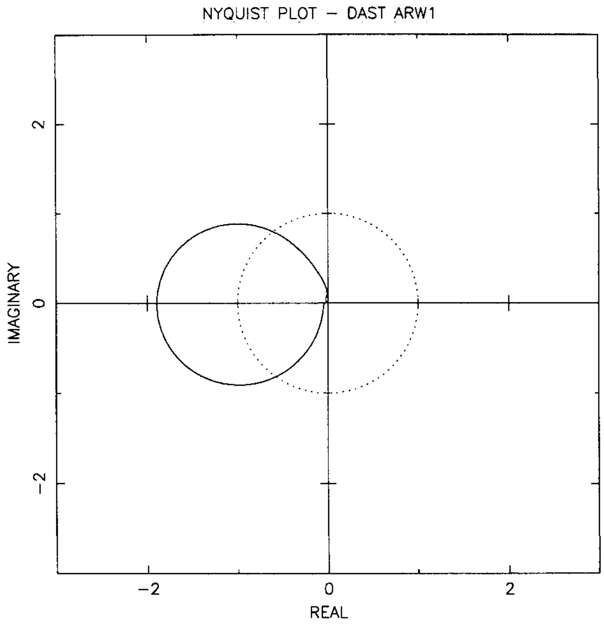

Fig. 2 Nyquist plot of plant with full-state controller (45th order).

Table 1 Effect on performance index J of omitting, in turn, a single x state, on the closed-loop plant-regulator system

Deleted regulator gain for x state	Performance index, J	Control deflection, δ	Control rate, $\dot{\delta}$	State of stability
None	1070.13	1.837	170.63	S
1	N/A	N/A	N/A	U
2	1124.44	1.885	173.02	S
3	1070.12	1.837	170.64	S
4	1070.23	1.837	170.57	S
5	1071.31	1.838	171.04	S
6	1070.18	1.837	170.67	S
7	1070.13	1.837	170.62	S
8	1070.13	1.837	170.64	S
9	1070.13	1.837	170.65	S
10	1070.13	1.837	170.64	S
11	N/A	N/A	N/A	U
12	1139.19	1.898	174.91	S
13	1070.13	1.837	170.64	S
14	1071.95	1.839	171.20	S
15	1070.16	1.836	170.51	S
16	1070.13	1.837	170.55	S
17	1070.13	1.837	170.64	S
18	1070.13	1.837	170.64	S
19	1070.13	1.837	170.64	S
20	1070.13	1.837	170.64	S
21	1072.10	1.844	166.67	S
22	1198.74	1.891	223.23	S
23	1070.13	1.837	170.64	S
24	1070.84	1.833	173.02	S
25	1071.17	1.834	173.87	S
26	1070.13	1.836	170.68	S
27	1070.13	1.837	170.68	S
28	1070.13	1.837	170.64	S
29	1070.13	1.837	170.64	S
30	1070.13	1.837	170.63	S
31	1070.31	1.834	171.39	S
32	1071.47	1.844	168.92	S
33	1070.13	1.837	170.64	S
34	1070.42	1.840	169.93	S
35	1070.16	1.836	170.46	S
36	1070.13	1.837	170.68	S
37	1070.13	1.837	170.63	S
38	1070.13	1.837	170.64	S
39	1070.13	1.837	170.64	S
40	1070.13	1.837	170.63	S
41	1123.08	1.884	188.52	S
42	1070.75	1.838	171.27	S
43	1074.17	1.844	179.89	S
44	1070.96	1.838	169.55	S
45	9152.85	5.470	185.07	S

Table 2 Summary of plant-regulator results in x space

Number of regulator feedback states	J	Q_D, psf	σ_{min}	δ, deg	$\dot{\delta}$, deg/s	Remarks
45	1070.13	203	1	1.837	170.64	Full-state feedback
43	9329.06	203	1	5.523	186.45	With no gust feedback
8	1074.88	202	0.829	1.846	169.87	σ_{min} at $w = 160$ rad/s
9	1073.37	202	0.961	1.844	169.30	σ_{min} at $w = 0.5$ rad/s

Table 3 Effect on performance index J of omitting, in turn, a single or double z state on the closed-loop transformed plant-regulator system

Deleted regulator gain for z state	Performance index, J	Control deflection, δ	Control rate, $\dot{\delta}$	State of stability
None	1070.13	1.837	170.64	S
1	N/A	N/A	N/A	U
2,3	N/A	N/A	N/A	U
4	1602.98	2.223	225.33	S
5,6	28305.93	9.614	418.08	S
7	1906.02	2.490	170.73	S
8,9	1070.19	1.838	169.88	S
10	1070.13	1.837	170.62	S
11,12	1070.13	1.837	170.64	S
13	1070.13	1.837	170.64	S
14,15	1070.13	1.836	170.77	S
16	1070.13	1.837	170.62	S
17,18	1070.13	1.837	170.64	S
19,20	1071.23	1.838	171.84	S
21	1070.13	1.837	170.59	S
22	1070.13	1.837	170.66	S
23,24	1070.13	1.836	170.70	S
25	1070.13	1.837	170.64	S
26	1070.13	1.837	170.64	S
27	1070.13	1.837	170.64	S
28,29	1070.16	1.837	169.97	S
30,31	1070.14	1.837	170.43	S
32	1070.15	1.838	170.38	S
33,34	1070.26	1.836	170.12	S
35,36	1070.13	1.836	170.47	S
37,38	1070.13	1.837	170.64	S
39,40	1070.13	1.837	170.59	S
41,42	1070.14	1.837	170.53	S
43,44	N/A	N/A	N/A	U
45	N/A	N/A	N/A	U

unmeasurable. However, this is irrelevant at this point because the purpose of the work presented in this section is to gain physical insight into the controller order reduction problem. Therefore, we proceed by setting each column of F, in turn, to zero and evaluating both J and the eigenvalues of the closed-loop system using Eqs. (4–6). The results of this exercise are summarized in Table 1. As can be seen, the feedback gains associated with many x states have essentially no effect on J, and therefore these states need neither be measured nor estimated. In most cases the system remains stable (denoted by S in the last column in Table 1). The omission of only two states (states 1 and 11) leads to instability (denoted by U in Table 1). Because the solution of the Lyapunov equation has no meaning whenever the system turns unstable, no values are given in the table for these cases and the N/A (not applicable) notation has been used instead. Based on the results shown in Table 1, the following eight x states will be considered important: 1, 2, 11, 12, 22, 41, 43, and 45. The inclusion of state 43 within the group of important states may be questionable, but this is irrelevant to our procedure because the purpose of this section is to obtain results with a small number of feedback states that are essentially identical to those relating to the full-state feedback system. Such results will be useful in making our point regarding the proposed order reduction procedure. The trade-offs between performance and a small number of feedback states may depend on the specific design and on the existing design constraints. The closed-loop results obtained with these eight states used for feedback are summarized in Table 2, together with results pertaining to other feedback conditions.

The root-locus plot for the full-state regulator feedback system is shown in Fig. 3. As can be seen, no flutter speed is encountered within the dynamic pressure range of 0–220 psf. Instead, a static divergence of one of the states is observed around $Q_D = 203$ psf. When the gust states are the only states not used for feedback, the value of J (Table 2) increases dramatically from $J = 1070.13$ for the full-state feedback to $J = 9329.06$. The value of δ increases from $\delta = 1.837$ deg for the full-state feedback case to $\delta = 5.523$ deg for the case where all states but the gust states are used for feedback. Clearly, the omission of the gust states from the regulator feedback results in greatly deteriorated performance, with absolutely no effect on stability.

The results for the previously determined eight-state regulator feedback system are presented in Table 2. These results are very close to the full-state feedback system, yielding $J = 1074.88$, $\delta = 1.846$ deg, and $\dot{\delta} = 169.87$ deg/s. The root-locus plot for this reduced-state feedback system, shown in Fig. 4, is essentially identical to the root-locus plot obtained for the full-state feedback case shown in Fig. 3. The only non-negligible difference is in the value of $\sigma_{min} = 0.829$ for the reduced-state feedback system, instead of $\sigma_{min} = 1$ for the full-state feedback system. The Nyquist plot for this reduced-state feedback system is shown in Fig. 5. The value of σ_{min} is obtained at a relatively high frequency, $w = 160$ rad/s. This frequency corresponds to the frequency of the fourth structural mode, which is represented by states 4 and 14. Table 1 shows that out of these two states, state 14 has a larger effect on J. Therefore, if we add state 14 to the previously determined eight states, giving a total of nine feedback states, we obtain results that are essentially identical to the full-state feedback case. The Nyquist plot for this nine-state feedback case is shown in Fig. 6. The shape of this Nyquist plot is very close to the circle of the full-state feedback system and yields a value of $\sigma_{\min} = 0.961$ instead of $\sigma_{\min} = 1.000$ for the full-state feedback case. Looking back at the results obtained so far, we can see that the important modes are the first two structural modes, which are represented by states 1, 2, 11, and 12. State 22 relates to an aerodynamic state and, of the 20 aerodynamic states, is the only important one. This is very interesting because the aerodynamic representation is otherwise responsible for the considerable augmentation of the plant's order. States 41 and 43 represent actuator states, and their being important comes as no surprise. Finally, state 45 represents one of the two gust states, and its importance to the values of J, δ, and $\dot{\delta}$ is naturally expected.

Analysis of the Plant-Controller System

At this point, a logical conclusion may be drawn that if only nine states are important for feedback, one needs to estimate only these nine states. Therefore, the observer equation (10) can be rearranged and truncated to include only those terms associated with the important states. If this is done, and the closed-loop performance of the plant-controller system is computed, very frustrating results are obtained. For our present example, the system is found to be unstable over a very wide range of dynamic pressures, thus raising severe questions about the validity of the proposed procedure. However, a closer look at Eq. (10) reveals that this observer equation is, in general, fully coupled. Therefore, while truncating this equation to include only the important $\hat{x}$ states, the reconstructed values of these important states are severely affected to the extent that the whole procedure becomes useless. Therefore, it is extremely important that during the process that leads to the determination of the important states, simultaneous attention should be given to the problem involving the reconstruction of these states. This leads to the transformation matrix T, which decouples the observer states, and to the mathematical formulation summarized in steps 3–7 of the proposed procedure.

At this point, step 3 of the proposed procedure is followed, leading to the transformed set of equations in terms of z states. This set of equations is given in steps 4–6 of the proposed procedure and is equivalent, in the z space, to Eqs. (4–6), which relate to the previously described x space. The only difference is that the observer equation in the $\hat{z}$ space is now in block diagonal form. The results obtained using steps 3–6 of the proposed procedure are presented in Table 3 and reveal the important z states in the plant-regulator feedback system. As can be seen from Table 3, the important z states are

Table 4 Summary of plant-controller results

Number of regulator feedback $\hat{z}$ states	J	Q_F, psf	$\sigma_{\min}$	δ, deg	$\dot{\delta}$, deg/s	Remarks
45	5484	191	0.886	4.22	183.0	Full-state feedback
10	5485	191	0.844	4.22	183.0	By residualization
10	5504	191	0.835	4.23	183.5	By truncation
0	1070	—	1	1.84	170.6	Full-state regulator

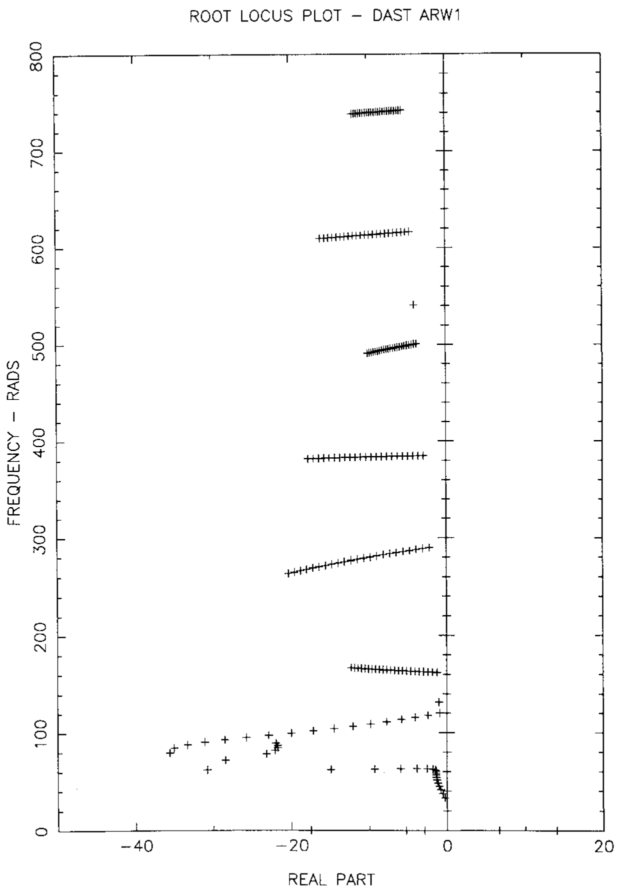

Fig. 3 Closed-loop root-locus plot of plant with full-state regulator feedback.

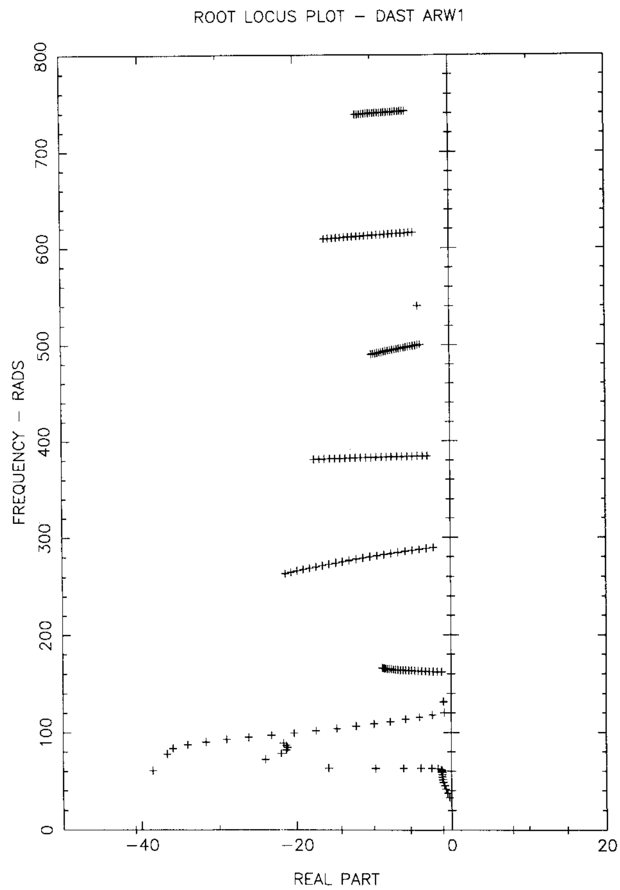

Fig. 4 Closed-loop root-locus plot of plant with eight-state regulator feedback.

Table 5 Actual effect on performance index J of omitting, in turn, a single observer state $\hat{x}_i$, on the plant-observer system and comparison with the cost components ψ_i using the CCA method

State $\hat{x}_i$	Component ψ_i in J	Value of J	State of stability	δ, rad	$\dot{\delta}$, rad/s	σ_{min}	Remarks
None	N/A	5484	S	4.2	183	0.886	$Q_F = 191$ at $\omega = 32$
1	4196.7	N/A	U	N/A	N/A	N/A	$Q_F = 13$ at $\omega = 54$
2	299.4	N/A	U	N/A	N/A	N/A	Unstable throughout
3	0.0	5484	S	4.2	183	0.886	$Q_F = 191$ at $\omega = 32$
5	−32.9	10689	S	5.9	255	0.246	Unstable at low Q
6	3.8	N/A	U	N/A	N/A	N/A	Unstable throughout
11	817.9	N/A	U	N/A	N/A	N/A	Unstable throughout
12	904.4	N/A	U	N/A	N/A	N/A	$Q_F = 90$ at $\omega = 22$
17	0.0	N/A	U	N/A	N/A	N/A	Unstable throughout
41	−318.0	5544	S	4.3	185	0.782	$Q_F = 191$ at $\omega = 32$
42	−17.5	N/A	U	N/A	N/A	N/A	Unstable throughout
45	−135.3	6037	S	4.4	178	0.933	$Q_F = 196$ at $\omega = 28$

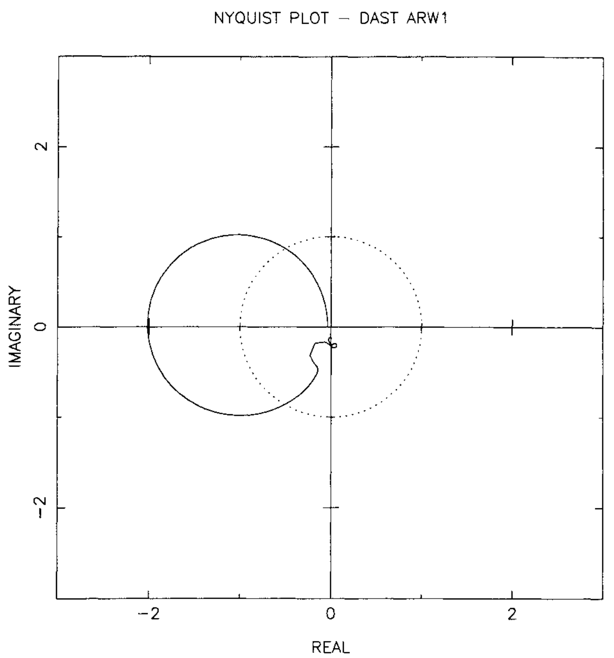

Fig. 5 Nyquist plot of plant with eight-state regulator feedback.

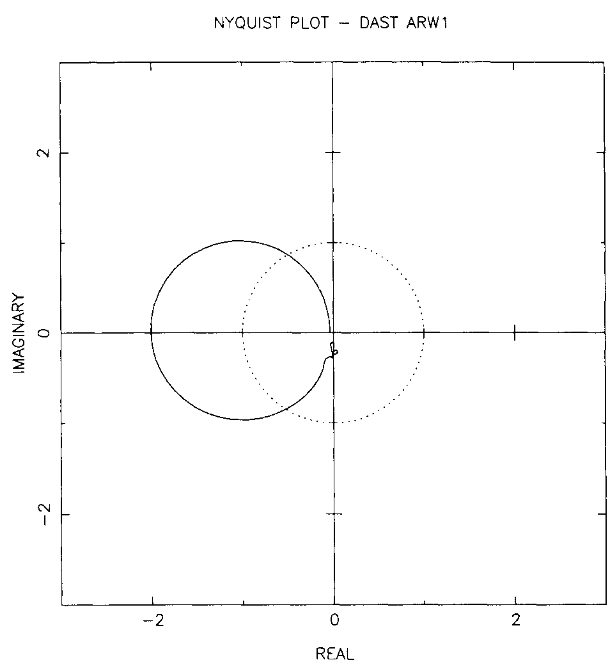

Fig. 6 Nyquist plot of plant with nine-state regulator feedback.

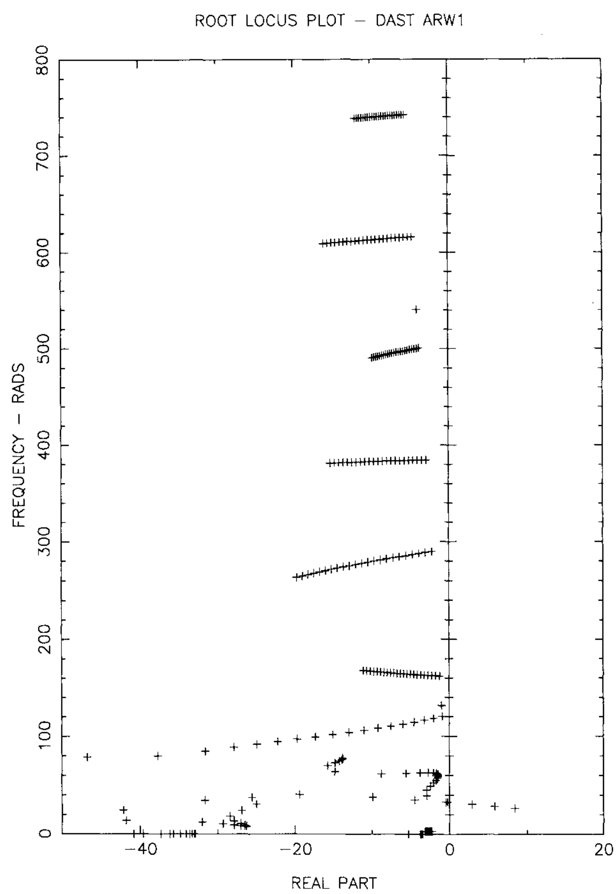

Fig. 7 Closed-loop root-locus plot of plant with 10th-order controller (residualized).

the following: 1–7 and 43–45. This yields a total of 10 important states. These important states can no longer be identified with any physical state, as was the case treated previously while working in the x space. Finally, we proceed with step 7 of our procedure, using the 10 important z states. The results are summarized in Table 4. The value of J for the 10th-order controller (with a total of 55 states) using residualization ($J = 5485$) is essentially the same as the value obtained using the full-order controller ($J = 5484$, with a total of 90 states). The control-surface activities and the flutter dynamic pressure are identical for both full- and reduced-order controller systems, with only small differences observed in the values of σ_{min}. For the full-order controller, $\sigma_{min} = 0.886$; for the 10th-order controller, $\sigma_{min} = 0.844$. The root-locus plot for the reduced-order controller is shown in Fig. 7. This is essentially identical to the root-locus plot for the full-order controller in

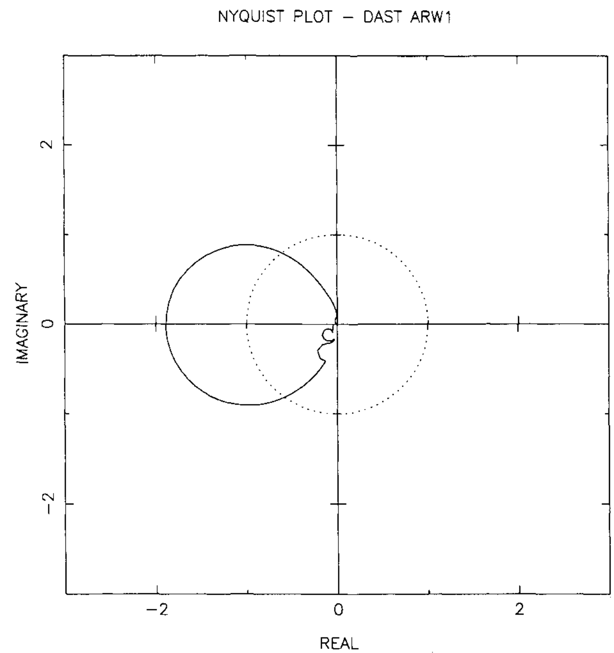

Fig. 8 Nyquist plot of plant with 10th-order controller (residualized).

Fig. 1. Similarly, the Nyquist plot for the reduced-order controller shown in Fig. 8 is very close to the corresponding plot for the full-order controller shown in Fig. 2.

Comparison with the Component Cost Analysis Method

The cost components ψ_i obtained using the CCA method are computed to illustrate some of the points mentioned earlier. These values are subsequently compared with the effects on J of reducing the plant-observer system by a single observer state, one at a time, using the $x,\hat{x}$ coordinates. Part of the results obtained are summarized in Table 5. The cost component metrics by which observer states with small values of ψ_i can be omitted to reduce the order of the controller is misleading. On one hand, observer states with large associated values of ψ_i are found to lead to instability (such as states 1, 2, 11, and 12), and observer states with small associated values of ψ_i are found to be unimportant (such as state 3). This is in complete agreement with the CCA method. On the other hand, however, states with very small associated values of ψ_i (some with 0 values), are found to have large effects on J, or are found to lead to instability (such as states 5, 6, 17, and 42). This is in complete disagreement with the CCA method. It can thus be seen that the CCA's basic assumption regarding controller order reduction does not appear to work.

Conclusions

The proposed method enables the order reduction of LQG-designed controllers with negligible effects on the closed-loop performance. The method indicates that the observer couplings between important and unimportant states greatly affect the reconstructed values of the important states. Therefore it is essential to first decouple the observer states using coordinate transformation, and only then to proceed with the determination of the important transformed states in the optimal plant-regulator system.

The example treated in the present paper consists of an aeroservoelastic problem with 45 states. Application of the proposed procedure to the 45th-order LQG-designed controller leads to a 10th-order controller that essentially maintains the performance properties of the full-order system.

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